Nonlinear dynamics of the cellular-automaton "game of Life"

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A statistical analysis of the "game of Life" due to Conway [Berlekamp, Conway, and Guy, *Winning Ways for Your Mathematical Plays* (Academic, New York, 1982), Vol. 2] is reported. The results are based on extensive computer simulations starting with uncorrelated distributions of live sites at $t = 0$. The number $n(s,t)$ of clusters of s live sites at time t, the mean cluster size $\bar{s}(t)$, and the diversity of sizes among other statistical functions are obtained. The dependence of the statistical functions with the initial density of live sites is examined. Several scaling relations as well as static and dynamic critical exponents are found.

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I. INTRODUCTION

The cellular-automaton (CA) game of Life (GL) introduced by Conway in 1970 [1] has been extensively studied in the last 20 years [1–6]. More recently some aspects on self-organized criticality [4], subcritical behavior [5], and time evolution of the density of living cells [6] in the GL have been examined. A CA is a lattice system in which the state of each lattice point is determined by local rules. More exactly, the GL is a totalistic class-4 CA. Totalistic means that the value of the cellular variable on a site depends only on the sum of the values of its neighbors at the previous time step, and not on their individual values [7]. Class 4 (CA) designates an automaton whose behavior may be determined only by explicit simulation [8]. Class-4 CA would be capable of universal computation: with particular initial states, their evolution could implement any finite algorithm. Universal computation has been proved for the GL [9]. The simple algorithm of the GL simulates the dynamical evolution of a society of living organisms. Processes such as growth, death, survival, and competition are included. The rules are fully deterministic; the state at time t gives a precise determination of the state at time $t + 1$. The reverse, however, is not true, i.e., the GL exhibits irreversibility [1]. Moreover, there are Life configurations that can only arise as the initial state, because they have no ancestors [1].

Although a lot of effort has been devoted to the understanding of the details of stable finite configurations [1], much less is known on the statistical properties of the GL. This is one of the main motivations for the present work. Here we report an extensive statistical analysis of the GL dynamics based on computer simulations. Previous interest in the GL has focused on the generation of complexity from specific initial configurations; indeed, the system has been suggested to mimic aspects of the emergence of complexity in nature [1]. The approach adopted here is different since it is inspired in statistical studies on other nonequilibrium (dissipative) processes such as fragmentation and consumption [10,11]. In our work a stochastic element is introduced at time $t = 0$ with a random distribution of live sites chosen with probability p. Some of the questions that are considered here include (i) the distribution function $n(s,t)$ of clusters of s live sites at time t, and (ii) the dependence of statistical functions describing the evolution of the GL with the initial conditions and with the size of the lattice L.

Although the GL is not related in any obvious way to specific physical or biological systems, there are various considerations which have led us to examine its properties. From the physical point of view we are interested in the nonequilibrium dynamics of the GL in connection with other nonequilibrium dynamics in dissipative processes as mentioned in the preceding paragraph. In particular we seek scaling laws in the GL dynamics. Theoretical and experimental investigations have shown that when certain spatially extended nonequilibrium systems are driven, they naturally evolve into a critical state characterized by spatial and temporal power laws [10,12–15]. The occurrence of such critical states in nonequilibrium systems is spontaneous; it does not require, as in equilibrium systems, the tuning of experimentally adjustable parameters to particular values or critical points. Several CA models have been shown both numerically and analytically to exhibit power laws under generic conditions [13,16]. The characterization and understanding of these scaling laws has become an important and surprisingly difficult problem. As examples we have diffusion-limited aggregation [12] and self-organized criticality [13] which are based on simple physical models, but whose detailed prediction of their power-law correlations has proven illusive. The simulations reported here demonstrate that similar power laws have evolved in GL. From the biological point of view, we are interested in the study of the complexity in the GL. Here complexity is evaluated by means of diversity. In this work we restrict ourselves to the study of cluster size diversity, i.e., the number of different sizes of clusters of live sites at a particular time [10]. In nature, the term diversity occurs in several contexts, as for example, in connection with forms [17], species [18], populations [19], and sizes [10]. The diversity of sizes is particularly appropriate for a quantitative numerical study. An alternative scaling relation for the diversity of sizes of clusters is reported for

the GL.

The structure of this paper is as follows. In Sec. II we give a brief description of our simulations. Results are discussed in Sec. III. We conclude in Sec. IV with a summary of the principal results.

II. SIMULATIONS

In this paper we study the statistics of the dynamics of the GL. The system is a two-dimensional square lattice of linear size L with the sites having two states, designated "live" or "dead." In our simulation L varied from 30 to 1100. We started at $t = 0$ with a random distribution of live sites with probability p. The rules for the evolution of the GL were implemented and at each time step the number $n(s,t)$ of clusters of s connected live sites (sites that are nearest neighbors of live sites) at time t was measured. The simulation was interrupted when the density and average coordination remained constant or displayed simple periodic oscillations for ten successive generations (*the stabilization regime*). We have considered extensively the dependence of the results on p. This probability in our work varies in intervals of 0.05 from $p = 0.05$ to 0.90. The moments of $n(s,t)$, diversity of sizes, and the average coordination of live sites are some statistical quantities which were studied as a function of time. Our results are averages varying from five experiments (for $L = 1100$) to 200 experiments (for $L = 30$). The rules of the GL invented by Conway [1] are the following.

(i) *Birth:* A site that is dead at time t becomes live at $t + 1$ only if exactly three of its eight neighbors were live at t.

(ii) *Death by overcrowding:* A site that is live at t and has four or more of its eight neighbors live at t will be dead by time $t + 1$.

(iii) *Death by exposure:* A live site that has only one live neighbor, or none at all, at time t, will also be dead at $t + 1$.

(iv) *Survival:* A site that was live at time t will remain live at $t + 1$ if and only if it had just two or three live neighbors at time t.

All births and deaths take place simultaneously. Neighbors are defined to be the eight sites surrounding the site under consideration, or in other terms, the first and second crystallographic nearest neighbors in a square planar lattice.

III. RESULTS

A basic quantity to be examined is the total number of live clusters at time t, $N(t) = \sum_s n(s,t)$, i.e., the zeroth momentum of $n(s,t)$. As explained in the beginning of the preceding section, a cluster of size s is a unit formed by s live sites connected within the first neighborhood. This criterion used to define a cluster is adopted in the statistics of the present work since the GL scaling functions are more clearly defined for this kind of neighbors. If other definitions of cluster are used, say extending out to second or third neighbors, slightly different results are obtained. However, we are not interested in this type of detail here. In particular, in Ref. [20] we make a brief discussion of the relation between the criteria of neighborhood and other statistical functions for the GL. The function $N(t)$ is shown in Fig. 1 for initial occupation probability 0.15 (a), 0.35 (b), 0.55 (c), and 0.75 (d) on a square lattice with $L = 300$. The plot refers to an average over ten similar experiments. For $0.15 \leq p < 0.75$ there is a scaling region along one to two decades in time where $N(t) \sim t^{-0.31 \pm 0.03}$. The inset shows in detail the behavior of $N(t)$ for $p = 0.35$. To obtain the smooth line of this inset we have used 50 bins to accommodate the data points. For $p < 0.15$ or $p \geq 0.75$ the behavior of $N(t)$ is similar to the curve d in Fig. 1, i.e., no scaling region is clearly identified. In general we have observed that the statistical functions describing the GL dynamics with the initial conditions considered in this article are divided in two groups. In a first group, for $0.15 \leq p < 0.75$, it is possible to identify a scaling region and well-defined critical ex-
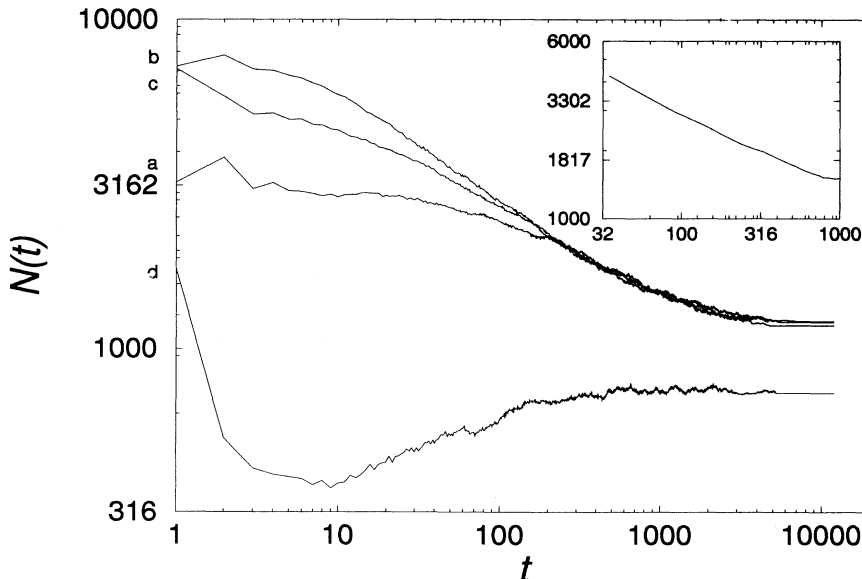

FIG. 1. Log-log plot of the total number of live clusters for the GL, $N(t)$, as a function of the time t. The initial occupation probabilities are $p = 0.15$ (a), 0.35 (b), 0.55 (c), and 0.75 (d). The data refer to simulations on square lattices of size $L = 300$, and the averages are over ten experiments. The inset shows $N(t)$ for $p = 0.35$ when 50 bins along the abscissa are used to accommodate the data points.
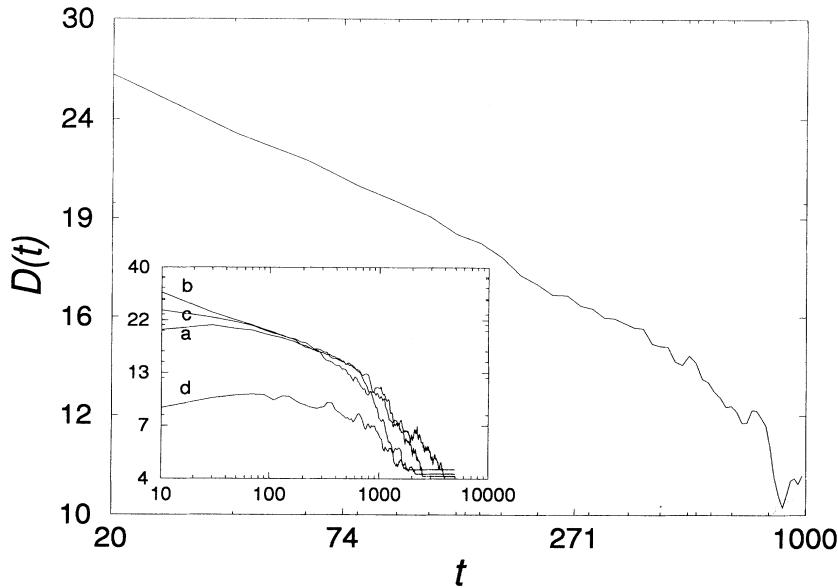
FIG. 2. Log-log plot of the time dependence of the diversity of live clusters, $D(t)$. $D(t)$ presents a scaling region in the same time interval of $N(t)$. The enlarged part shows $D(t)$ for $p = 0.35$ when 45 bins along the t axis are used. The inset exhibits the evolution of $D(t)$ for $p = 0.15$ (a), 0.35 (b), 0.55 (c), and 0.75 (d), without the use of bins.

ponents. In a second group, including small value of p ($p < 0.15$) or large values of p ($p \geq 0.75$), it is not possible to identify any scaling region. Bagnoli, Rechtman, and Ruffo [6] recently studied the dependence of the asymptotic density of live sites ρ_∞ versus the initial density $\rho_0 \equiv p$ in the GL. It is interesting to observe that these authors have found that for $0.15 < p < 0.75$ ρ_∞ does not depend on ρ_0 and assumes a constant value near $\rho_\infty = 0.028$. Moreover, they found that for $p < 0.15$ or $p > 0.75$, ρ_∞ varied rapidly with ρ_0. So, in our simulations we have found that (i) the occurrence of temporal scaling is apparently connected with the region where ρ_∞ is not dependent on ρ_0 (or p), and (ii) the absence of temporal scaling is connected with those regions where ρ_∞ depends on ρ_0.

The diversity $D(t)$ (number of different sizes of live clusters at time t irrespective of shape) is shown in Fig. 2 for experiments on lattices with $L = 150$. The inset of Fig. 2 presents $D(t)$ for $p = 0.15$ (a), 0.35 (b), 0.55 (c), and 0.75 (d). As observed in Fig. 1, for cases (a), (b), and (c), $D(t)$ has a power-law scaling of the form $D(t) \sim t^{-0.24 \pm 0.02}$, but for $p = 0.75$ (d) the time scaling disappears. In Fig. 2 we illustrate the quality of the power-law behavior for $D(t)$ for $p = 0.35$. The (enlarged) smooth line of Fig. 2 corresponding to $p = 0.35$ is obtained by distributing the data points in 45 bins along the t axis. In comparison with other nonlinear dissipative dynamical processes generating distributions of clusters [10,11], the GL presents wider intervals for the dynamical scaling even if small lattice sizes are used.
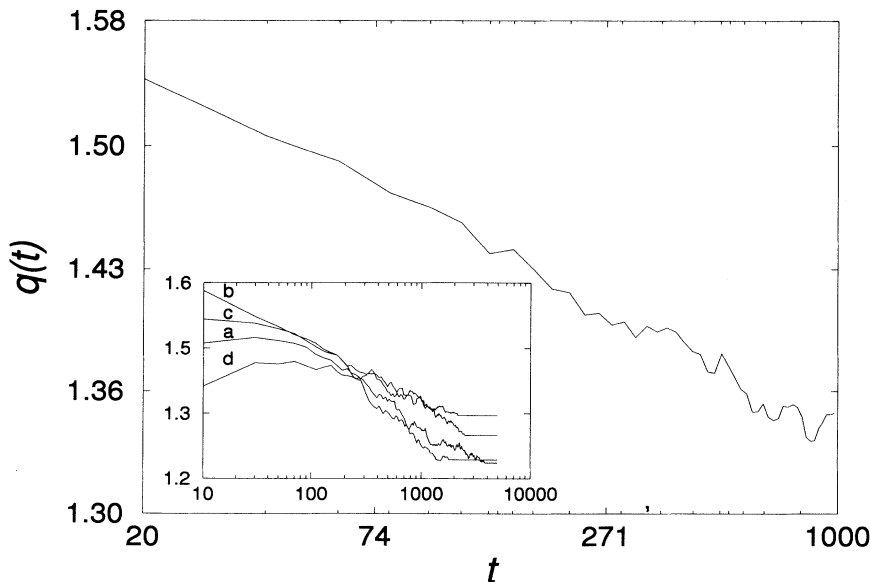

FIG. 3. Log-log plot of the time evolution of the average coordination number $q(t)$ for the live sites. The enlarged plot refers to $p = 0.35$ (50 bins) and the other parameters are the same as those in Fig. 2. $q(t) \sim t^{-0.04 \pm 0.005}$ (Sec. III, third paragraph). The inset shows $q(t)$ for the values of p considered in Figs. 1 and 2 (without bins).
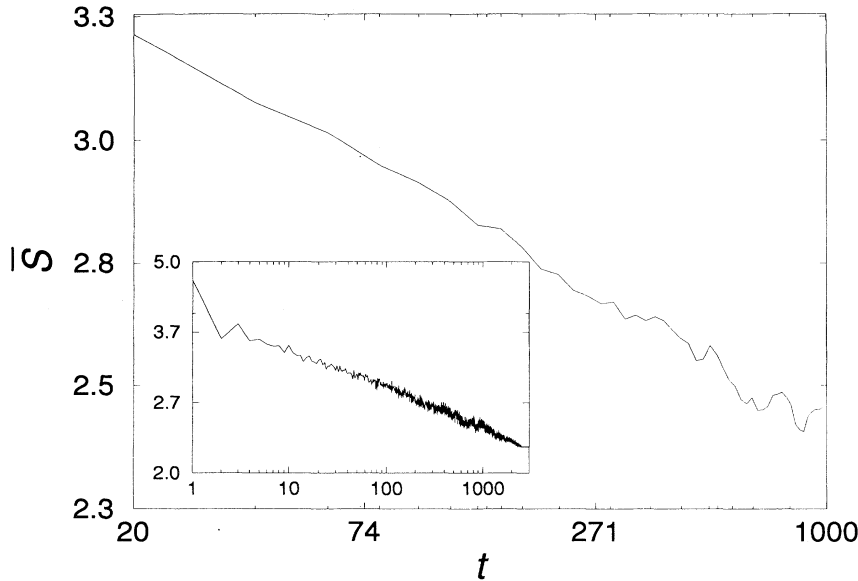
FIG. 4. Log-log plot of the mean cluster size $\bar{s}(t)$ in the scaling region for $p = 0.35$. $s(t) \sim t^{-0.08 \pm 0.01}$ (Sec. III, third paragraph). In the enlarged part we show $\bar{s}(t)$ when 50 bins are used to accommodate the data points. The inset exhibits $\bar{s}(t)$ without the use of bins.

The process of formation of live clusters in the GL involves a continuous change in connectivity. The average coordination number $q(t)$ for the live sites gives a direct measure of this change [11]. As shown in Fig. 3, $q(t)$ decays as the power law $q \sim t^{-0.04 \pm 0.005}$ along one to two decades in t. The enlarged part of this figure refers to an ensemble of ten experiments with lattices of size $L = 150$ and $p = 0.35$. In this case we have used 50 bins along the t axis to accommodate the data points. The inset shows the average coordination number for $p = 0.15$ (a), 0.35 (b), 0.55 (c), and 0.75 (d). As can be seen from this inset q scales with t with the same exponent, irrespective of p, even for large values of the initial occupation probability ($p \leq 0.75$). The power-law decay of $q(t)$ is apparently a characteristic of the GL since $q(t)$ in general does not ex-

hibit this kind of behavior for other nonlinear cluster dynamics [10,15,21,22]. The mean cluster size $\bar{s}(t)$ is another quantity of interest. In the GL $\bar{s}(t)$ decreases with t monotonically in a large interval as $\bar{s} \sim t^{-0.08 \pm 0.01}$ irrespective of p. For $L = 150$ this time interval covers more than two decades in t as shown in Fig. 4 for $p = 0.35$. The inset of Fig. 4 shows the complete variation of $\bar{s}$ from $t = 1$ to the stabilization regime. In the enlarged part of Fig. 4 the data points are distributed in 50 bins along the t axis. From the power law for $q(t)$ and $\bar{s}(t)$ we conclude that these two quantities are related by the simple expression $\bar{s}(t) \sim q(t)^2$ in the scaling region.

The average number of clusters of s live sites in the stabilization regime, $n_\infty(s)$, is shown in Fig. 5 for a lattice with $L = 300$ and for initial occupation probabilities 0.15
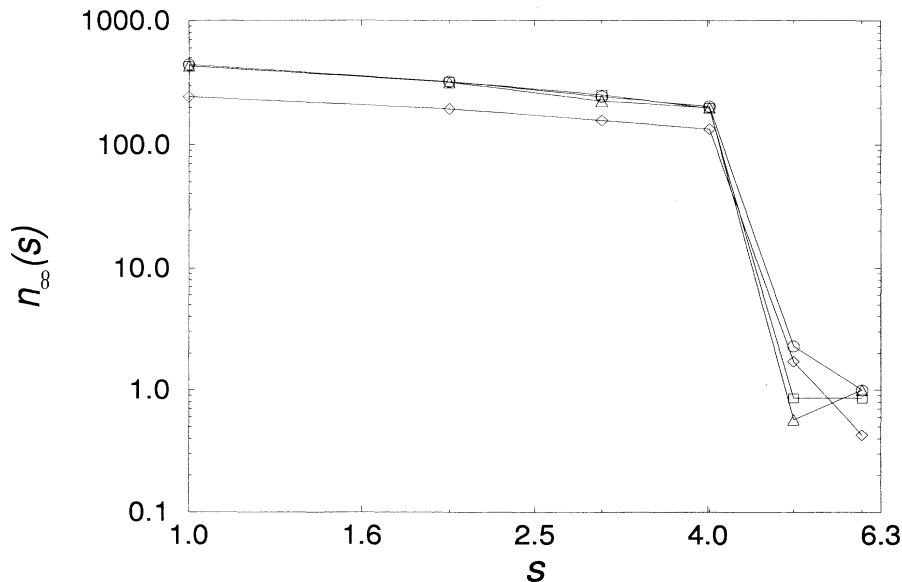

FIG. 5. Log-log plot of the average number of clusters of s live sites in the stabilization regime, $n_\infty(s)$, on a lattice with 300^2 sites, for $p = 0.15$ ($\square$), 0.35 ($\bigcirc$), (0.55 ($\triangle$), and 0.75 ($\Diamond$). A sudden transition occurs in this case near $s = 4$.
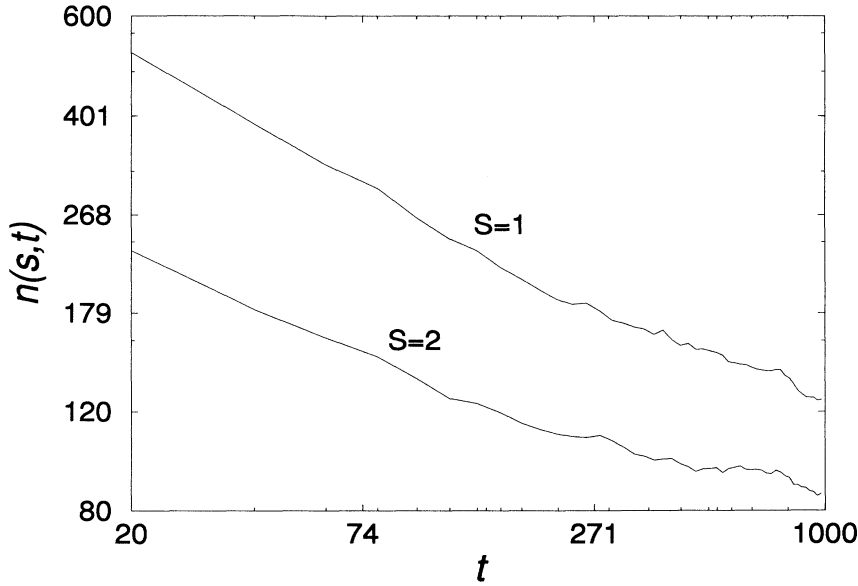
FIG. 6. Log-log plot of the number of live clusters with one and two live sites as a function of time, for the GL on a lattice with $L = 300$ (see text, Sec. III, fourth paragraph).

(open square), 0.35 (open circle), 0.55 (open triangle), and 0.75 (open diamond). For this size the stabilization regime begins at about $t = 4000$. The cluster distribution in Fig. 5 refers to $t = 10\,000$. As can be seen from Fig. 5, $n_\infty(s)$ undergoes a sudden transition near $s = 4$ in the stabilization regime (this plot refers to an average over ten similar experiments).

In Fig. 6 we show the number of live clusters of size $s = 1$, and $s = 2$, as a function of t for $p = 0.35$. The t axis in this figure was divided in 45 bins. We obtain that $n(s=1, t) \sim t^{w_1}$, $w_1 = 0.33 \pm 0.05$, and $n(s=2,5) \sim t^{w_2}$, $w_2 = 0.23 \pm 0.05$ with both critical exponents independent of p, provided $0.15 \leq p < 0.75$. For $p < 0.15$ or $p \geq 0.75$, however, it is impossible to find a dynamic scaling regime for both $n(s=1, t)$ and $n(s=2, t)$. Possibly, $n(s,t) \sim t^{-w}$,

with $w \simeq 0.3$ for any (small) value of s in the limit $L \to \infty$ and that the difference observed in the exponents ($w_1 \neq w_2$) is a finite size effect. Furthermore, this exponent w would be the same exponent that scales N and t.

We have seen in the preceding paragraphs that the GL presents a scaling region before the stabilization regime where $N(t) \sim t^{-0.31 \pm 0.03}$ and $\bar{s}(t) \sim t^{-0.08 \pm 0.01}$. This means that the total population (or mass) of live sites evolves as $m(t) = N(t)\bar{s}(t) \sim t^{-0.39 \pm 0.04}$. We cannot forget that this power-law relation for $m(t)$ is valid only in the interval $0.15 \leq p < 0.75$. For $p < 0.15$ or $p > 0.75$ there is no clear scaling for $N(t)$ or $m(t)$. Figure 7 exhibits the dependence of $m(t)$ for the GL on lattices with size $L = 300$ and probabilities $p = 0.15$ (a), 0.35 (b), 0.55 (c), and 0.75 (d). All curves represent averages over ten
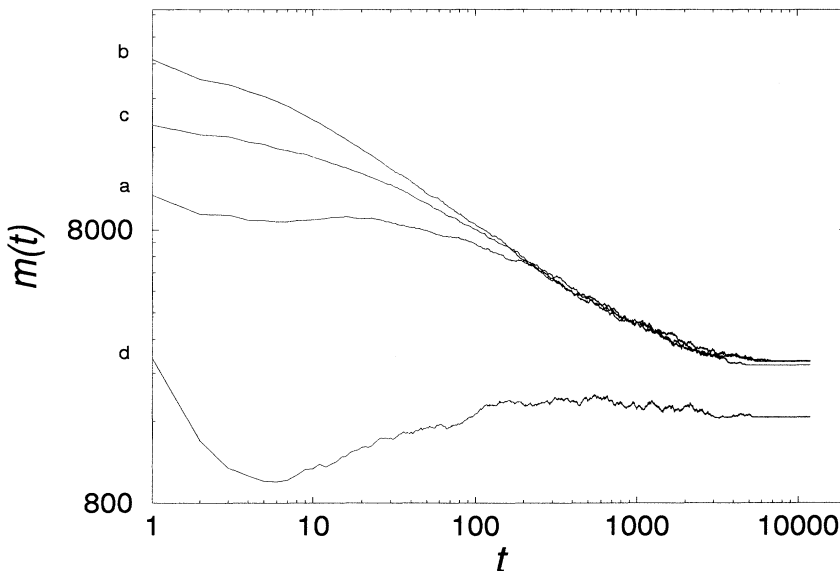

FIG. 7. Log-log plot of the total population of live sites, $m(t)$, on a lattice with 300^2 sites for $p = 0.15$ (a), 0.35 (b), 0.55 (c), and 0.75 (d). In the first three situations we have $m \sim t^{-0.39 \pm 0.04}$ (Sec. III, fifth paragraph).
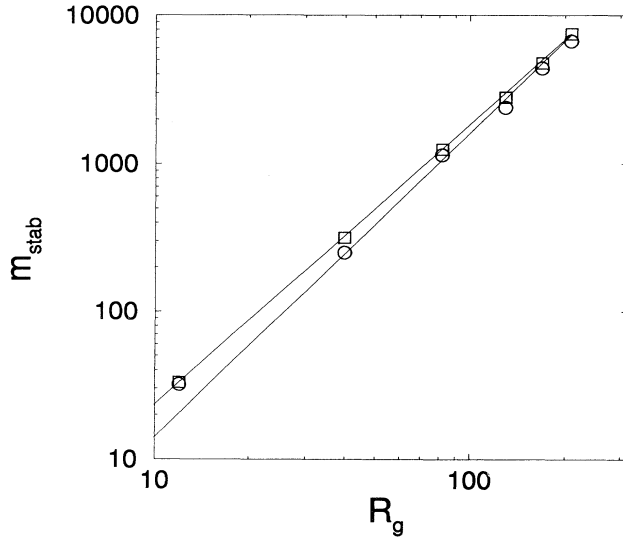
FIG. 8. Log-log plot of the number m_{stab} of live sites as a function of the radius of gyration R_g for $p = 0.35$ ($\square$) and $p = 0.70$ ($\circ$). The six values of R_g used in this plot correspond to simulations on lattices of sizes $L = 30, 100, 200, 300, 400$, and 500. Each data point represents an average over five similar experiments (see text, Sec. III, sixth paragraph).

similar experiments. Here again, as in Figs. 1, 2, and 6, there is indeed evidence of a nontrivial scaling regime for the density of live sites, before the onset of stabilization.

Finally we have investigated how the population of live sites in the stabilization regime, m_{stab}, scales with the radius of gyration R_g. This quantity is defined as $R_g = \sqrt{\sum_{i,j}(\mathbf{r}_i - \mathbf{r}_j)^2 / 2m^2}$, where $\mathbf{r}_i$ denotes the position of the live site i from the center of gravity and m is the total population considered. As shown in Fig. 8, $m_{stab} \sim R_g^{\delta_{stab}}$, with $\delta_{stab} = 1.88$ for $p = 0.35$, and $\delta_{stab} = 2.0$ for $p = 0.70$. Our overall estimate is that m_{stab} scales with the radius of gyration R_g as $R_g^{1.94 \pm 0.06}$ irrespective of the value of the initial occupation probability. These results suggest that Life distributes itself (within the statistical uncertainties) on a disconnected set of dimension $\delta = 2$. The plots in Fig. 8 refer to simulations in lattices of sizes $L = 30, 100, 200, 300, 400$, and 500, and each point

represents an average on five similar experiments. Recently Bak, Chen, and Creutz [4] reported that the number of live sites in the GL increases with the distance r from a given live site as $r^{1.7}$, i.e., the GL distributes itself on a fractal set of dimension 1.7. The results of this paragraph do not agree with this conclusion. We believe that the value 1.7 found by Bak, Chen, and Creutz is a consequence of the relatively small lattices (150×150) used in Ref. [4]. Anyway, computer simulations on larger systems are needed to decide this specific question.

IV. CONCLUSIONS

This paper presents the first extensive statistical description of the famous Conway game of Life [1] considering as initial states the random occupation with live sites with probability p on lattices of size L up to 1100. For initial occupation probabilities satisfying $0.15 \leq p < 0.75$, each one of the different statistical functions φ describing the dynamics of the GL may be divided in general in three intervals: First, a region extending from $t = 0$ to $t \simeq L^{1/2}$ presenting large fluctuations in φ; second, a scaling region characterized by a power-law dependence between φ and t, from $t \sim L^{1/2}$ to $t \sim L^{4/3}$; and finally the "steady state" or stabilization region (the SOC state of Bak, Chen, and Creutz [4]) extending from $t \sim L^{4/3}$ to infinite and characterized by small fluctuations of φ around some average value φ_0. These results are obtained from extensive numerical simulations on lattices with different values of L. The critical exponents obtained in the scaling region are robust and do not depend on p, for $0.15 \leq p < 0.75$. The dynamics of the GL examined here is compared with other nonequilibrium dissipative dynamics of physical, chemical, and biological interest recently studied [10,11,15,22]. For low initial occupation ($p < 0.15$) or high density occupation ($p \geq 0.75$) the scaling region disappears, and now the domain for large fluctuations of φ extends from $t = 0$ to $t \sim L^{4/3}$. This region is followed by the "steady state" characterized by small fluctuations around an average value.

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